Student Solutions Manual and Study Guide

Chapters 1 & 2 Preview

for

Numerical Analysis

9th EDITION

Richard L. Burden
Youngstown State University

J. Douglas Faires
Youngstown State University

Prepared by

Richard L. Burden
Youngstown State University

J. Douglas Faires
Youngstown State University

BROOKS/COLE
CENGAGE Learning™

Australia • Brazil • Japan • Korea • Mexico • Singapore • Spain • United Kingdom • United States

BROOKS/COLE
CENGAGE Learning

ISBN-13: 978-0-538-73563-6
ISBN-10: 0-538-73563-5

Brooks/Cole
20 Channel Center Street
Boston, MA 02210
USA

Cengage Learning is a leading provider of customized learning solutions with office locations around the globe, including Singapore, the United Kingdom, Australia, Mexico, Brazil, and Japan. Locate your local office at: **international.cengage.com/region**

Cengage Learning products are represented in Canada by Nelson Education, Ltd.

For your course and learning solutions, visit **academic.cengage.com**

Purchase any of our products at your local college store or at our preferred online store **www.ichapters.com**

Contents

Preface

This Student Solutions Manual and Study Guide for *Numerical Analysis*, Ninth Edition, by Burden and Faires contains representative exercises that have been worked out in detail for all the techniques discussed in the book. Particular attention was paid to ensure that the exercises solved in the Guide are those requiring insight into the theory and methods discussed in the book. Although the answers to the odd exercises are also in the back of the book, the results listed in this Study Guide generally go well beyond those in the book.

For this edition we have added a number of exercises to the text that involve the use of a computer algebra system (CAS). We chose Maple as our standard CAS, because their *NumericalAnalysis* package parallels the algorithms in this book. However, any of the common computer algebra systems, such Mathematica, MATLAB, and the public domain system, Sage, can be used with satisfaction. In our recent teaching of the course we have found that students understood the concepts better when they worked through the algorithms step-by-step, but let a computer algebra system do the tedious computation.

It has been our practice to include structured algorithms in our Numerical Analysis book for all the techniques discussed in the text. The algorithms are given in a form that can be coded in any appropriate programming language, by students with even a minimal amount of programming expertise.

At the website for the book,

http://www.math.ysu.edu/∼faires/Numerical-Analysis/

you will find code for all the algorithms written in the programming languages FORTRAN, Pascal, C, Java. You will also find code in the form of worksheets for the computer algebra systems, Maple, MATLAB, and Mathematica. For this edition we have rewritten all the Maple programs to reflect the *NumericalAnalysis* package and the numerous changes that have been made to this system.

The website contains additional information about the book, and will be updated regularly to reflect any modifications that might be made. For example, we will place there any errata we are aware of, as well as responses to questions from users of the book concerning interpretations of the exercises and appropriate applications of the techniques.

We hope our Guide helps you with your study of Numerical Analysis. If you have any suggestions for improvements that can be incorporated into future editions of the book or the supplements, we would be most grateful to receive your comments. We can be most easily contacted by electronic mail at the addresses listed below.

Youngstown State University — Richard L. Burden, burden@math.ysu.edu

August 14, 2010 — J. Douglas Faires, faires@math.ysu.edu

Mathematical Preliminaries

Exercise Set 1.1, page 14

1. d. Show that the equation $x - (\ln x)^x = 0$ has at least one solution in the interval $[4, 5]$.

SOLUTION: It is not possible to algebraically solve for the solution x, but this is not required in the problem, we must show only that a solution exists. Let

$$f(x) = x - (\ln x)^x = x - \exp(x(\ln(\ln x))).$$

Since f is continuous on $[4, 5]$ with $f(4) \approx 0.3066$ and $f(5) \approx -5.799$, the Intermediate Value Theorem 1.11 implies that a number x must exist in $(4, 5)$ with $0 = f(x) = x - (\ln x)^x$.

2. c. Find intervals that contain a solution to the equation $x^3 - 2x^2 - 4x + 3 = 0$.

SOLUTION: Let $f(x) = x^3 - 2x^2 - 4x + 3$. The critical points of f occur when

$$0 = f'(x) = 3x^2 - 4x - 4 = (3x + 2)(x - 2);$$

that is, when $x = -\frac{2}{3}$ and $x = 2$. Relative maximum and minimum values of f can occur only at these values. There are at most three solutions to $f(x) = 0$, because $f(x)$ is a polynomial of degree three. Since $f(-2) = -5$ and $f\left(-\frac{2}{3}\right) \approx 4.48$; $f(0) = 3$ and $f(1) = -2$; and $f(2) = -5$ and $f(4) = 19$; solutions lie in the intervals $[-2, -2/3]$, $[0, 1]$, and $[2, 4]$.

4. a. Find $\max\limits_{0 \le x \le 1} |f(x)|$ when $f(x) = (2 - e^x + 2x)/3$.

SOLUTION: First note that $f'(x) = (-e^x + 2)/3$, so the only critical point of f occurs at $x = \ln 2$, which lies in the interval $[0, 1]$. The maximum for $|f(x)|$ must consequently be

$$\max\{|f(0)|, |f(\ln 2)|, |f(1)|\} = \max\{1/3, (2\ln 2)/3, (4 - e)/3\} = (2\ln 2)/3.$$

5. Use the Intermediate Value Theorem 1.11 and Rolle's Theorem 1.7 to show that the graph of $f(x) = x^3 + 2x + k$ crosses the x-axis exactly once, regardless of the value of the constant k.

SOLUTION: For $x < 0$,we have $f(x) < 2x + k < 0$, provided that $x < -\frac{1}{2}k$. Similarly, for $x > 0$, we have $f(x) > 2x + k > 0$, provided that $x > -\frac{1}{2}k$. By Theorem 1.11, there exists a number c with $f(c) = 0$.

If $f(c) = 0$ and $f(c') = 0$ for some $c' \neq c$, then by Theorem 1.7, there exists a number p between c and c' with $f'(p) = 0$. However, $f'(x) = 3x^2 + 2 > 0$ for all x. This gives a contradiction to the statement that $f(c) = 0$ and $f(c') = 0$ for some $c' \neq c$. Hence there is exactly one number c with $f(c) = 0$.

9. Find the second Taylor polynomial for $f(x) = e^x \cos x$ about $x_0 = 0$, and:

a. Use $P_2(0.5)$ to approximate $f(0.5)$, find an upper bound for $|f(0.5) - P_2(0.5)|$, and compare this to the actual error.

b. Find a bound for the error $|f(x) - P_2(x)|$, for x in $[0, 1]$.

c. Approximate $\int_0^1 f(x)\,dx$ using $\int_0^1 P_2(x)\,dx$.

d. Find an upper bound for the error in part (c).

SOLUTION: Since

$$f'(x) = e^x(\cos x - \sin x), \quad f''(x) = -2e^x(\sin x), \quad \text{and} \quad f'''(x) = -2e^x(\sin x + \cos x),$$

we have $f(0) = 1$, $f'(0) = 1$, and $f''(0) = 0$. So

$$P_2(x) = 1 + x \quad \text{and} \quad R_2(x) = \frac{-2e^\xi(\sin\xi + \cos\xi)}{3!}x^3.$$

a. We have $P_2(0.5) = 1 + 0.5 = 1.5$ and

$$|f(0.5) - P_2(0.5)| \le \max_{\xi\in[0.0.5]} \left|\frac{-2e^\xi(\sin\xi + \cos\xi)}{3!}(0.5)^2\right| \le \frac{1}{3}(0.5)^2 \max_{\xi\in[0,0.5]} |e^\xi(\sin\xi + \cos\xi)|.$$

To maximize this quantity on $[0, 0.5]$, first note that $D_x e^x(\sin x + \cos x) = 2e^x \cos x > 0$, for all x in $[0, 0.5]$. This implies that the maximum and minimum values of $e^x(\sin x + \cos x)$ on $[0, 0.5]$ occur at the endpoints of the interval, and

$$e^0(\sin 0 + \cos 0) = 1 < e^{0.5}(\sin 0.5 + \cos 0.5) \approx 2.24.$$

Hence

$$|f(0.5) - P_2(0.5)| \le \frac{1}{3}(0.5)^3(2.24) \approx 0.0932.$$

b. A similar analysis to that in part (a) gives, for all $x \in [0, 1]$,

$$|f(x) - P_2(x)| \le \frac{1}{3}(1.0)^3 e^1(\sin 1 + \cos 1) \approx 1.252.$$

c.

$$\int_0^1 f(x)\,dx \approx \int_0^1 1 + x\,dx = \left[x + \frac{x^2}{2}\right]_0^1 = \frac{3}{2}.$$

d. From part (b),

$$\int_0^1 |R_2(x)|\,dx \le \int_0^1 \frac{1}{3}e^1(\cos 1 + \sin 1)x^3\,dx = \int_0^1 1.252x^3\,dx = 0.313.$$

Since

$$\int_0^1 e^x \cos x\,dx = \left[\frac{e^x}{2}(\cos x + \sin x)\right]_0^1 = \frac{e}{2}(\cos 1 + \sin 1) - \frac{1}{2}(1 + 0) \approx 1.378,$$

the actual error is $|1.378 - 1.5| \approx 0.12$.

14. Use the error term of a Taylor polynomial to estimate the error involved in using $\sin x \approx x$ to approximate $\sin 1^\circ$.

SOLUTION: First we need to convert the degree measure for the sine function to radians. We have $180^\circ = \pi$ radians, so $1^\circ = \frac{\pi}{180}$ radians. Since $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, and $f'''(x) = -\cos x$, we have $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$. The approximation $\sin x \approx x$ is given by $f(x) \approx P_2(x)$ and $R_2(x) = -\frac{\cos \xi}{3!}x^3$. If we use the bound $|\cos \xi| \leq 1$, then

$$\left|\sin\left(\frac{\pi}{180}\right) - \frac{\pi}{180}\right| = \left|R_2\left(\frac{\pi}{180}\right)\right| = \left|\frac{-\cos\xi}{3!}\left(\frac{\pi}{180}\right)^3\right| \leq 8.86 \times 10^{-7}.$$

16. Let $f(x) = e^{x/2} \sin \frac{x}{3}$.

a. Use Maple to determine the third Maclaurin polynomial $P_3(x)$.

b. Find $f^{(4)}(x)$ and bound the error $|f(x) - P_3(x)|$ on $[0, 1]$.

SOLUTION: **a.** Define $f(x)$ by

$f := exp\left(\frac{x}{2}\right) \cdot sin\left(\frac{x}{3}\right)$

$$f := e^{(1/2)x} \sin\left(\frac{1}{3}x\right)$$

Then find the first three terms of the Taylor series with

$g := taylor(f, x = 0, 4)$

$$g := \frac{1}{3}x + \frac{1}{6}x^2 + \frac{23}{648}x^3 + O\left(x^4\right)$$

Extract the third Maclaurin polynomial with

$p3 := convert(g, polynom)$

$$p3 := \frac{1}{3}x + \frac{1}{6}x^2 + \frac{23}{648}x^3$$

b. Determine the fourth derivative.

$f4 := diff(f, x, x, x, x)$

$$f4 := -\frac{119}{1296}e^{(1/2x)} \sin\left(\frac{1}{3}x\right) + \frac{5}{54}e^{(1/2x)} \cos\left(\frac{1}{3}x\right)$$

Find the fifth derivative.

$f5 := diff(f4, x)$

$$f5 := -\frac{199}{2592}e^{(1/2x)} \sin\left(\frac{1}{3}x\right) + \frac{61}{3888}e^{(1/2x)} \cos\left(\frac{1}{3}x\right)$$

See if the fourth derivative has any critical points in $[0, 1]$.

$p := fsolve(f5 = 0, x, 0..1)$

$$p := .6047389076$$

The extreme values of the fourth derivative will occur at $x = 0$, 1, or p.

$c1 := evalf(subs(x = p, f4))$

$$c1 := .09787176213$$

$c2 := evalf(subs(x = 0, f4))$

$$c2 := .09259259259$$

$c3 := evalf(subs(x = 1, f4))$

$$c3 := .09472344463$$

The maximum absolute value of $f^{(4)}(x)$ is c_1 and the error is given by

$error := c1/24$

$$error := .004077990089$$

24. In Example 3 it is stated that x we have $|\sin x| \leq |x|$. Use the following to verify this statement.

a. Show that for all $x \geq 0$ the function $f(x) = x - \sin x$ is non-decreasing, which implies that $\sin x \leq x$ with equality only when $x = 0$.

b. Use the fact that the sine function is odd to reach the conclusion.

SOLUTION: First observe that for $f(x) = x - \sin x$ we have $f'(x) = 1 - \cos x \geq 0$, because $-1 \leq \cos x \leq 1$ for all values of x. Also, the statement clearly holds when $|x| \geq \pi$, because $|\sin x| \leq 1$.

a. The observation implies that $f(x)$ is non-decreasing for all values of x, and in particular that $f(x) > f(0) = 0$ when $x > 0$. Hence for $x \geq 0$, we have $x \geq \sin x$, and when $0 \leq x \leq \pi$, we have $|\sin x| = \sin x \leq x = |x|$.

b. When $-\pi < x < 0$, we have $\pi \geq -x > 0$. Since $\sin x$ is an odd function, the fact (from part (a)) that $\sin(-x) \leq (-x)$ implies that $|\sin x| = -\sin x \leq -x = |x|$.

As a consequence, for all real numbers x we have $|\sin x| \leq |x|$.

28. Suppose $f \in C[a, b]$, and that x_1 and x_2 are in $[a, b]$.

a. Show that a number ξ exists between x_1 and x_2 with

$$f(\xi) = \frac{f(x_1) + f(x_2)}{2} = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

b. Suppose that c_1 and c_2 are positive constants. Show that a number ξ exists between x_1 and x_2 with

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

c. Give an example to show that the result in part (b) does not necessarily hold when c_1 and c_2 have opposite signs with $c_1 \neq -c_2$.

SOLUTION:

a. The number

$$\frac{1}{2}(f(x_1) + f(x_2))$$

is the average of $f(x_1)$ and $f(x_2)$, so it lies between these two values of f. By the Intermediate Value Theorem 1.11 there exist a number ξ between x_1 and x_2 with

$$f(\xi) = \frac{1}{2}(f(x_1) + f(x_2)) = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

b. Let $m = \min\{f(x_1), f(x_2)\}$ and $M = \max\{f(x_1), f(x_2)\}$. Then $m \le f(x_1) \le M$ and $m \le f(x_2) \le M$, so

$$c_1 m \le c_1 f(x_1) \le c_1 M \quad \text{and} \quad c_2 m \le c_2 f(x_2) \le c_2 M.$$

Thus

$$(c_1 + c_2)m \le c_1 f(x_1) + c_2 f(x_2) \le (c_1 + c_2)M$$

and

$$m \le \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} \le M.$$

By the Intermediate Value Theorem 1.11 applied to the interval with endpoints x_1 and x_2, there exists a number ξ between x_1 and x_2 for which

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

c. Let $f(x) = x^2 + 1$, $x_1 = 0$, $x_2 = 1$, $c_1 = 2$, and $c_2 = -1$. Then $f(x) > 0$ for all values of x, but

$$\frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} = \frac{2(1) - 1(2)}{2 - 1} = 0.$$

Exercise Set 1.2, page 28

2. **c.** Find the largest interval in which p^* must lie to approximate $\sqrt{2}$ with relative error at most 10^{-4}.

SOLUTION: We need

$$\frac{|p^* - \sqrt{2}|}{|\sqrt{2}|} \le 10^{-4}, \quad \text{so} \quad \left|p^* - \sqrt{2}\right| \le \sqrt{2} \times 10^{-4};$$

that is,

$$-\sqrt{2} \times 10^{-4} \le p^* - \sqrt{2} \le \sqrt{2} \times 10^{-4}.$$

This implies that p^* must be in the interval $\left(\sqrt{2}(0.9999), \sqrt{2}(1.0001)\right)$.

5. **e.** Use three-digit rounding arithmetic to compute

$$\frac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4},$$

and determine the absolute and relative errors.

SOLUTION: Using three-digit rounding arithmetic gives $\frac{13}{14} = 0.929$, $\frac{6}{7} = 0.857$, and $e = 2.72$. So

$$\frac{13}{14} - \frac{6}{7} = 0.0720 \quad \text{and} \quad 2e - 5.4 = 5.44 - 5.40 = 0.0400.$$

Hence

$$\frac{\frac{13}{14}-\frac{6}{7}}{2e-5.4}=\frac{0.0720}{0.0400}=1.80.$$

The correct value is approximately 1.954, so the absolute and relative errors to three digits are

$$|1.80-1.954|=0.154 \quad \text{and} \quad \frac{|1.80-1.954|}{1.954}=0.0788,$$

respectively.

7. e. Repeat Exercise 5(e) using three-digit chopping arithmetic.

SOLUTION: Using three-digit chopping arithmetic gives $\frac{13}{14}=0.928$, $\frac{6}{7}=0.857$, and $e=2.71$. So

$$\frac{13}{14}-\frac{6}{7}=0.0710 \quad \text{and} \quad 2e-5.4=5.42-5.40=0.0200.$$

Hence

$$\frac{\frac{13}{14}-\frac{6}{7}}{2e-5.4}=\frac{0.0710}{0.0200}=3.55.$$

The correct value is approximately 1.954, so the absolute and relative errors to three digits are

$$|3.55-1.954|=1.60, \quad \text{and} \quad \frac{|3.55-1.954|}{1.954}=0.817,$$

respectively. The results in Exercise 5(e) were considerably better.

9. a. Use the first three terms of the Maclaurin series for the arctangent function to approximate $\pi=4\left[\arctan\frac{1}{2}+\arctan\frac{1}{3}\right]$, and determine the absolute and relative errors.

SOLUTION: Let $P(x)=x-\frac{1}{3}x^3+\frac{1}{5}x^5$. Then $P\left(\frac{1}{2}\right)=0.464583$ and $P\left(\frac{1}{3}\right)=0.3218107$, so

$$\pi=4\left[\arctan\frac{1}{2}+\arctan\frac{1}{3}\right]\approx 3.145576.$$

The absolute and relative errors are, respectively,

$$|\pi-3.145576|\approx 3.983\times 10^{-3} \quad \text{and} \quad \frac{|\pi-3.145576|}{|\pi|}\approx 1.268\times 10^{-3}.$$

12. Let

$$f(x)=\frac{e^x-e^{-x}}{x}.$$

a. Find $\lim_{x\to 0} f(x)$.

b. Use three-digit rounding arithmetic to evaluate $f(0.1)$.

c. Replace each exponential function with its third Maclaurin polynomial and repeat part (b).

SOLUTION: **a.** Since $\lim_{x\to 0} e^x-e^{-x}=1-1=0$ and $\lim_{x\to 0} x=0$, we can use L'Hospitals Rule to give

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{1}=\frac{1+1}{1}=2.$$

b. With three-digit rounding arithmetic we have $e^{0.100} = 1.11$ and $e^{-0.100} = 0.905$, so

$$f(0.100) = \frac{1.11 - 0.905}{0.100} = \frac{0.205}{0.100} = 2.05.$$

c. The third Maclaurin polynomials give

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \quad \text{and} \quad e^{-x} \approx 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3,$$

so

$$f(x) \approx \frac{\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right) - \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right)}{x} = \frac{2x + \frac{1}{3}x^3}{x} = 2 + \frac{1}{3}x^2.$$

Thus, with three-digit rounding, we have

$$f(0.100) \approx 2 + \frac{1}{3}(0.100)^2 = 2 + (0.333)(0.001) = 2.00 + 0.000333 = 2.00.$$

15. c. Find the decimal equivalent of the floating-point machine number

$$0 \quad 01111111111 \quad 0101001100000000000000000000000000000000000000000000.$$

SOLUTION: This binary machine number is the decimal number

$$\begin{aligned} &+ 2^{1023-1023}\left(1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^8\right) \\ &= 2^0\left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{128} + \frac{1}{256}\right) = 1 + \frac{83}{256} = 1.32421875. \end{aligned}$$

16. c. Find the decimal equivalents of the next largest and next smallest floating-point machine number to

$$0 \quad 01111111111 \quad 0101001100000000000000000000000000000000000000000000.$$

SOLUTION: The next smallest machine number is

$$\begin{aligned} &0 \quad 01111111111 \quad 0101001011111111111111111111111111111111111111111111 \\ &=1.32421875 - 2^{1023-1023}\left(2^{-52}\right) \\ &=1.3242187499999997779553950749686919152736663818359375, \end{aligned}$$

and next largest machine number is

$$\begin{aligned} &0 \quad 01111111111 \quad 0101001100000000000000000000000000000000000000000001 \\ &=1.32421875 + 2^{1023-1023}\left(2^{-52}\right) \\ &=1.3242187500000002220446049250313080847263336181640625. \end{aligned}$$

21. a. Show that the polynomial nesting technique can be used to evaluate

$$f(x) = 1.01e^{4x} - 4.62e^{3x} - 3.11e^{2x} + 12.2e^x - 1.99.$$

b. Use three-digit rounding arithmetic and the formula given in the statement of part (a) to evaluate $f(1.53)$.

c. Redo the calculations in part (b) using the nesting form of $f(x)$ that was found in part (a).

d. Compare the approximations in parts (b) and (c).

SOLUTION: **a.** Since $e^{nx} = (e^x)^n$, we can write

$$f(x) = ((((1.01)e^x - 4.62)\, e^x - 3.11)\, e^x + 12.2)\, e^x - 1.99.$$

b. Using $e^{1.53} = 4.62$ and three-digit rounding gives $e^{2(1.53)} = (4.62)^2 = 21.3$, $e^{3(1.53)} = (4.62)^2(4.62) = (21.3)(4.62) = 98.4$, and $e^{4(1.53)} = (98.4)(4.62) = 455$. So

$$\begin{aligned} f(1.53) &= 1.01(455) - 4.62(98.4) - 3.11(21.3) + 12.2(4.62) - 1.99 \\ &= 460 - 455 - 66.2 + 56.4 - 1.99 \\ &= 5.00 - 66.2 + 56.4 - 1.99 \\ &= -61.2 + 56.4 - 1.99 = -4.80 - 1.99 = -6.79. \end{aligned}$$

c. We have

$$\begin{aligned} f(1.53) &= (((1.01)4.62 - 4.62)4.62 - 3.11)4.62 + 12.2)4.62 - 1.99 \\ &= (((4.67 - 4.62)4.62 - 3.11)4.62 + 12.2)4.62 - 1.99 \\ &= ((0.231 - 3.11)4.62 + 12.2)4.62 - 1.99 \\ &= (-13.3 + 12.2)4.62 - 1.99 = -7.07. \end{aligned}$$

d. The exact result is 7.61, so the absolute errors in parts (b) and (c) are, respectively, $|-6.79 + 7.61| = 0.82$ and $|-7.07 + 7.61| = 0.54$. The relative errors are, respectively, 0.108 and 0.0710.

24. Suppose that $fl(y)$ is a k-digit rounding approximation to y. Show that

$$\left|\frac{y - fl(y)}{y}\right| \le 0.5 \times 10^{-k+1}.$$

SOLUTION: We will consider the solution in two cases, first when $d_{k+1} \le 5$, and then when $d_{k+1} > 5$.

When $d_{k+1} \le 5$, we have

$$\left|\frac{y - fl(y)}{y}\right| = \frac{0.d_{k+1}\ldots \times 10^{n-k}}{0.d_1 \ldots \times 10^n} \le \frac{0.5 \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

When $d_{k+1} > 5$, we have

$$\left|\frac{y - fl(y)}{y}\right| = \frac{(1 - 0.d_{k+1}\ldots) \times 10^{n-k}}{0.d_1 \ldots \times 10^n} < \frac{(1 - 0.5) \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

Hence the inequality holds in all situations.

28. Show that both sets of data given in the opening application for this chapter can give values of T that are consistent with the ideal gas law.

SOLUTION: For the initial data, we have

$$0.995 \leq P \leq 1.005, \quad 0.0995 \leq V \leq 0.1005,$$

$$0.082055 \leq R \leq 0.082065, \quad \text{and} \quad 0.004195 \leq N \leq 0.004205.$$

This implies that

$$287.61 \leq T \leq 293.42.$$

Since 15° Celsius = 288.16 kelvin, we are within the bound. When P is doubled and V is halved,

$$1.99 \leq P \leq 2.01 \quad \text{and} \quad 0.0497 \leq V \leq 0.0503,$$

so

$$286.61 \leq T \leq 293.72.$$

Since 19° Celsius = 292.16 kelvin, we are again within the bound. In either case it is possible that the actual temperature is 290.15 kelvin $=$ 17° Celsius.

Exercise Set 1.3, page 39

3. a. Determine the number n of terms of the series

$$\arctan x = \lim_{n\to\infty} P_n(x) = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{x^{2i-1}}{(2i-1)}$$

that are required to ensure that $|4P_n(1) - \pi| < 10^{-3}$.

b. How many terms are required to ensure the 10^{-10} accuracy needed for an approximation to π?

SOLUTION: **a.** Since the terms of the series

$$\pi = 4\arctan 1 = 4\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{2i-1}$$

alternate in sign, the error produced by truncating the series at any term is less than the magnitude of the next term. To ensure significant accuracy, we need to choose n so that

$$\frac{4}{2(n+1)-1} < 10^{-3} \quad \text{or} \quad 4000 < 2n+1.$$

So $n \geq 2000$.

b. In this case, we need

$$\frac{4}{2(n+1)-1} < 10^{-10} \quad \text{or} \quad n > 20{,}000{,}000{,}000.$$

Clearly, a more rapidly convergent method is needed for this approximation.

5. Another formula for computing π can be deduced from the identity

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}.$$

Determine the number of terms that must be summed to ensure an approximation to π to within 10^{-3}.

SOLUTION: The identity implies that

$$\pi = 4\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{5^{2i-1}(2i-1)} - \sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{239^{2i-1}(2i-1)}$$

The second sum is much smaller than the first sum. So we need to determine the minimal value of i so that the $i+1$st term of the first sum is less than 10^{-3}. We have

$$i := 1 : \frac{4}{5^1(1)} = \frac{4}{5}, \quad i = 2 : \frac{4}{5^3(3)} = \frac{4}{375} \quad \text{and} \quad i = 3 : \frac{4}{5^5(5)} = \frac{4}{15625} = 2.56 \times 10^{-4}.$$

So 3 terms are sufficient.

8. a. How many calculations are needed to determine a sum of the form

$$\sum_{i=1}^{n}\sum_{j=1}^{i} a_i b_j?$$

b. Re-express the series in a way that will reduce the number of calculations needed to determine this sum.

SOLUTION: **a.** For each i, the inner sum $\sum_{j=1}^{i} a_i b_j$ requires i multiplications and $i-1$ additions, for a total of

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{multiplications} \quad \text{and} \quad \sum_{i=1}^{n} i - 1 = \frac{n(n+1)}{2} - n \quad \text{additions.}$$

Once the n inner sums are computed, $n-1$ additions are required for the final sum.

The final total is:

$$\frac{n(n+1)}{2} \quad \text{multiplications} \quad \text{and} \quad \frac{(n+2)(n-1)}{2} \quad \text{additions.}$$

b. By rewriting the sum as

$$\sum_{i=1}^{n}\sum_{j=1}^{i} a_i b_j = \sum_{i=1}^{n} a_i \sum_{j=1}^{i} b_j,$$

we can significantly reduce the amount of calculation. For each i, we now need $i-1$ additions to sum b_j's for a total of

$$\sum_{i=1}^{n} i - 1 = \frac{n(n+1)}{2} - n \quad \text{additions.}$$

Once the b_j's are summed, we need n multiplications by the a_i's, followed by $n-1$ additions of the products.

The total additions by this method is still $\frac{1}{2}(n+2)(n-1)$, but the number of multiplications has been reduced from $\frac{1}{2}n(n+1)$ to n.

10. Devise an algorithm to compute the real roots of a quadratic equation in the most efficient manner.

SOLUTION: The following algorithm uses the most effective formula for computing the roots of a quadratic equation.

INPUT A, B, C.
OUTPUT x_1, x_2.

Step 1 If $A = 0$ then
    if $B = 0$ then OUTPUT ('NO SOLUTIONS');
        STOP.
    else set $x_1 = -C/B$;
        OUTPUT ('ONE SOLUTION',x_1);
        STOP.

Step 2 Set $D = B^2 - 4AC$.

Step 3 If $D = 0$ then set $x_1 = -B/(2A)$;
    OUTPUT ('MULTIPLE ROOTS', x_1);
    STOP.

Step 4 If $D < 0$ then set
    $b = \sqrt{-D}/(2A)$;
    $a = -B/(2A)$;
    OUTPUT ('COMPLEX CONJUGATE ROOTS');
    $x_1 = a + bi$;
    $x_2 = a - bi$;
    OUTPUT (x_1, x_2);
    STOP.

Step 5 If $B \geq 0$ then set
    $d = B + \sqrt{D}$;
    $x_1 = -2C/d$;
    $x_2 = -d/(2A)$
  else set
    $d = -B + \sqrt{D}$;
    $x_1 = d/(2A)$;
    $x_2 = 2C/d$.

Step 6 OUTPUT (x_1, x_2);
    STOP.

15. Suppose that as x approaches zero,

$$F_1(x) = L_1 + O\left(x^{\alpha}\right) \quad \text{and} \quad F_2(x) = L_2 + O\left(x^{\beta}\right).$$

Let c_1 and c_2 be nonzero constants, and define

$$F(x) = c_1F_1(x) + c_2F_2(x) \quad \text{and} \quad G(x) = F_1(c_1x) + F_2(c_2x).$$

Show that if $\gamma = \text{minimum}\ \{\alpha, \beta\}$, then as x approaches zero,

a. $F(x) = c_1L_1 + c_2L_2 + O\left(x^{\gamma}\right)$

b. $G(x) = L_1 + L_2 + O\left(x^\gamma\right)$

SOLUTION: Suppose for sufficiently small $|x|$ we have positive constants k_1 and k_2 independent of x, for which

$$|F_1(x) - L_1| \le K_1|x|^\alpha \quad \text{and} \quad |F_2(x) - L_2| \le K_2|x|^\beta.$$

Let $c = \max\left(|c_1|, |c_2|, 1\right)$, $K = \max\left(K_1, K_2\right)$, and $\delta = \max\left(\alpha, \beta\right)$.

a. We have

$$\begin{aligned} |F(x) - c_1L_1 - c_2L_2| =& |c_1(F_1(x) - L_1) + c_2(F_2(x) - L_2)| \\ \le& |c_1|K_1|x|^\alpha + |c_2|K_2|x|^\beta \\ \le& cK\left(|x|^\alpha + |x|^\beta\right) \\ \le& cK|x|^\gamma\left(1 + |x|^{\delta-\gamma}\right) \le K|x|^\gamma, \end{aligned}$$

for sufficiently small $|x|$. Thus, $F(x) = c_1L_1 + c_2L_2 + O\left(x^\gamma\right)$.

b. We have

$$\begin{aligned} |G(x) - L_1 - L_2| =& |F_1(c_1x) + F_2(c_2x) - L_1 - L_2| \\ \le& K_1|c_1x|^\alpha + K_2|c_2x|^\beta \\ \le& Kc^\delta\left(|x|^\alpha + |x|^\beta\right) \\ \le& Kc^\delta|x|^\gamma\left(1 + |x|^{\delta-\gamma}\right) \le K''|x|^\gamma, \end{aligned}$$

for sufficiently small $|x|$. Thus, $G(x) = L_1 + L_2 + O\left(x^\gamma\right)$.

16. Consider the Fibonacci sequence defined by $F_0 = 1$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$, if $n \ge 0$. Define $x_n = F_{n+1}/F_n$. Assuming that $\lim_{n\to\infty} x_n = x$ converges, show that the limit is the golden ratio: $x = \left(1 + \sqrt{5}\right)/2$.

SOLUTION: Since

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x \quad \text{and} \quad x_{n+1} = 1 + \frac{1}{x_n},$$

we have

$$x = 1 + \frac{1}{x}, \quad \text{which implies that} \quad x^2 - x - 1 = 0.$$

The only positive solution to this quadratic equation is $x = \left(1 + \sqrt{5}\right)/2$.

17. The Fibonacci sequence also satisfies the equation

$$F_n \equiv \tilde{F}_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

a. Write a Maple procedure to calculate F_{100}.

b. Use Maple with the default value of *Digits* followed by *evalf* to calculate $\tilde{F}_{100}$.

c. Why is the result from part (a) more accurate than the result from part (b)?

d. Why is the result from part (b) obtained more rapidly than the result from part (a)?

e. What results when you use the command *simplify* instead of *evalf* to compute $\tilde{F}_{100}$?

SOLUTION: **a.** To save space we will show the Maple output for each step in one line. Maple would produce this output on separate lines. The procedure for calculating the terms of the sequence are:

$n := 98; f := 1; s := 1$

$$n := 98 \quad f := 1 \quad s := 1$$

for i *from* 1 *to* n *do*
$l := f + s; f := s; s := l; od :$

$$\begin{aligned}
&l := 2 \quad f := 1 \quad s := 2\\
&l := 3 \quad f := 2 \quad s := 3\\
&\quad\vdots\\
&l := 218922995834555169026 \quad f := 135301852344706746049 \quad s := 218922995834555169026\\
&l := 354224848179261915075
\end{aligned}$$

b. We have

$F100 := \dfrac{1}{sqrt(5)}\left(\left(\dfrac{(1+sqrt(5))}{2}\right)^{100} - \left(\dfrac{1-sqrt(5)}{2}\right)^{100}\right)$

$$F100 := \frac{1}{\sqrt{5}}\left(\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{100} - \left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^{100}\right)$$

evalf$(F100)$

$$0.3542248538 \times 10^{21}$$

c. The result in part (a) is computed using exact integer arithmetic, and the result in part (b) is computed using ten-digit rounding arithmetic.

d. The result in part (a) required traversing a loop 98 times.

e. The result is the same as the result in part (a).

Solutions of Equations of One Variable

Exercise Set 2.1, page 54

1. Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

SOLUTION: Using the Bisection method gives $a_1 = 0$ and $b_1 = 1$, so $f(a_1) = -1$ and $f(b_1) = 0.45970$. We have

$$p_1 = \frac{1}{2}(a_1 + b_1) = \frac{1}{2} \quad \text{and} \quad f(p_1) = -0.17048 < 0.$$

Since $f(a_1) < 0$ and $f(p_1) < 0$, we assign $a_2 = p_1 = 0.5$ and $b_2 = b_1 = 1$. Thus

$$f(a_2) = -0.17048 < 0, \quad f(b_2) = 0.45970 > 0, \quad \text{and} \quad p_2 = \frac{1}{2}(a_2 + b_2) = 0.75.$$

Since $f(p_2) = 0.13434 > 0$, we have $a_3 = 0.5$; $b_3 = p_3 = 0.75$ so that

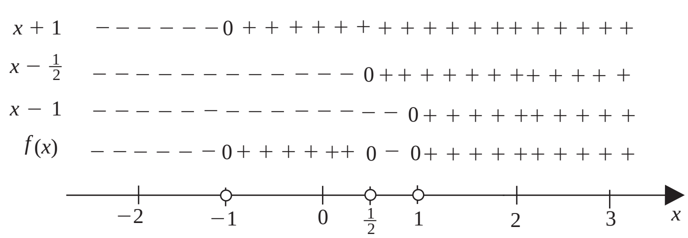
$$p_3 = \frac{1}{2}(a_3 + b_3) = 0.625.$$

2. a. Let $f(x) = 3(x+1)\left(x - \frac{1}{2}\right)(x-1)$. Use the Bisection method on the interval $[-2, 1.5]$ to find p_3.

SOLUTION: Since

$$f(x) = 3(x+1)\left(x - \frac{1}{2}\right)(x-1),$$

we have the following sign graph for $f(x)$:



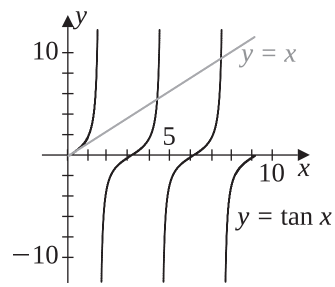
Thus, $a_1 = -2$, with $f(a_1) < 0$, and $b_1 = 1.5$, with $f(b_1) > 0$. Since $p_1 = -\frac{1}{4}$, we have $f(p_1) > 0$. We assign $a_2 = -2$, with $f(a_2) < 0$, and $b_2 = -\frac{1}{4}$, with $f(b_2) > 0$. Thus, $p_2 = -1.125$ and $f(p_2) < 0$. Hence, we assign $a_3 = p_2 = -1.125$ and $b_3 = -0.25$. Then $p_3 = -0.6875$.

8. **a.** Sketch the graphs of $y = x$ and $y = \tan x$.

b. Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = \tan x$.

SOLUTION:

a. The graphs of $y = x$ and $y = \tan x$ are shown in the figure. From the graph it appears that the graphs cross near $x = 4.5$.



b. Because $g(x) = x - \tan x$ has

$$g(4.4) \approx 1.303 > 0 \quad \text{and} \quad g(4.6) \approx -4.260 < 0,$$

the fact that g is continuous on $[4.4, 4.6]$ gives us a reasonable interval to start the bisection process. Using Algorithm 2.1 gives $p_{16} = 4.4934143$, which is accurate to within 10^{-5}.

11. Let $f(x) = (x+2)(x+1)x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge for the following intervals?

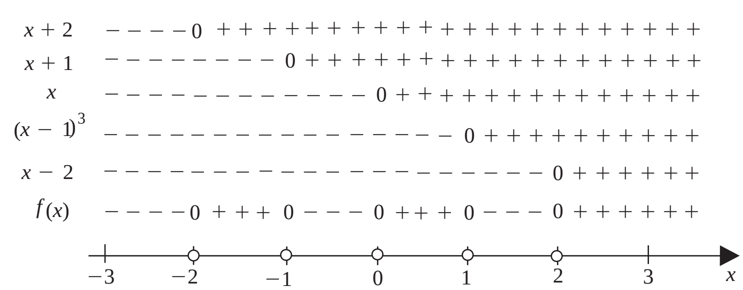
a. $[-3, 2.5]$

c. $[-1.75, 1.5]$

SOLUTION: Since

$$f(x) = (x+2)(x+1)x(x-1)^3(x-2),$$

we have the following sign graph for $f(x)$.



a. The interval $[-3, 2.5]$ contains all 5 zeros of f. For $a_1 = -3$, with $f(a_1) < 0$, and $b_1 = 2.5$, with $f(b_1) > 0$, we have $p_1 = (-3 + 2.5)/2 = -0.25$, so $f(p_1) < 0$. Thus we assign $a_2 = p_1 = -0.25$, with $f(a_2) < 0$, and $b_2 = b_1 = 2.5$, with $f(b_1) > 0$.

Hence $p_2 = (-0.25 + 2.5)/2 = 1.125$ and $f(p_2) < 0$. Then we assign $a_3 = 1.125$, with $f(a_3) < 0$, and $b_3 = 2.5$, with $f(b_3) > 0$. Since $[1.125, 2.5]$ contains only the zero 2, the method converges to 2.

c. The interval $[-1.75, 1.5]$ contains the zeros -1, 0, 1. For $a_1 = -1.75$, with $f(a_1) > 0$, and $b_1 = 1.5$, with $f(b_1) < 0$, we have $p_1 = (-1.75 + 1.5)/2 = -0.125$ and $f(p_1) < 0$. Then we assign $a_2 = a_1 = -1.75$, with $f(a_1) > 0$, and $b_2 = p_1 = -0.125$, with $f(b_2) < 0$. Since $[-1.75, -0.125]$ contains only the zero -1, the method converges to -1.

12. Use the Bisection Algorithm to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}.

SOLUTION: The function defined by $f(x) = x^2 - 3$ has $\sqrt{3}$ as its only positive zero. Applying the Bisection method to this function on the interval $[1, 2]$ gives $\sqrt{3} \approx p_{14} = 1.7320$. Using a smaller starting interval would decrease the number of iterations that are required.

14. Use Theorem 2.1 to find a bound for the number of iterations needed to approximate a solution to the equation $x^3 + x - 4 = 0$ on the interval $[1, 4]$ to an accuracy of 10^{-3}.

SOLUTION: First note that the particular equation plays no part in finding the bound; all that is needed is the interval and the accuracy requirement. To find an approximation that is accurate to within 10^{-3}, we need to determine the number of iterations n so that

$$|p - p_n| < \frac{b-a}{2^n} = \frac{4-1}{2^n} < 0.001; \quad \text{that is,} \quad 3 \times 10^3 < 2^n.$$

As a consequence, a bound for the number of iterations is $n \geq 12$. Applying the Bisection Algorithm gives $p_{12} = 1.3787$.

17. Define the sequence $\{p_n\}$ by $p_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$. Show that $\displaystyle\lim_{n\to\infty}(p_n - p_{n-1}) = 0$, even though the sequence $\{p_n\}$ diverges.

SOLUTION: Since $p_n - p_{n-1} = 1/n$, we have $\lim_{n\to\infty}(p_n - p_{n-1}) = 0$. However, p_n is the nth partial sum of the divergent *harmonic* series. The harmonic series is the classic example of a series whose terms go to zero, but not rapidly enough to produce a convergent series. There are many proofs of the divergence of this series, any calculus text should give at least two. One proof will simply analyze the partial sums of the series and another is based on the Integral Test.

The point of the problem is not the fact that this particular sequence diverges, it is that a test for an approximate solution to a root based on the condition that $|p_n - p_{n-1}|$ is small should always be suspect. Consecutive terms of a sequence might be close to each other, but not sufficiently close to the actual solution you are seeking.

19. A trough of water of length $L = 10$ feet has a cross section in the shape of a semicircle with radius $r = 1$ foot. When filled with water to within a distance h of the top, the volume $V = 12.4$ ft^3 of the water is given by the formula

$$12.4 = 10\left[0.5\pi - \arcsin h - h\left(1 - h^2\right)^{1/2}\right]$$

Determine the depth of the water to within 0.01 feet.

SOLUTION: Applying the Bisection Algorithm on the interval $[0,1]$ to the function

$$f(h) = 12.4 - 10\left[0.5\pi - \arcsin h - h\left(1-h^2\right)^{1/2}\right]$$

gives $h \approx p_{13} = 0.1617$, so the depth is $r - h \approx 1 - 0.1617 = 0.8383$ feet.

Exercise Set 2.2, page 64

3. The following methods are proposed to compute $21^{1/3}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a. $p_n = \dfrac{20p_{n-1} + 21/p_{n-1}^2}{21}$

b. $p_n = p_{n-1} - \dfrac{p_{n-1}^3 - 21}{3p_{n-1}^2}$

c. $p_n = p_{n-1} - \dfrac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$

d. $p_n = \left(\dfrac{21}{p_{n-1}}\right)^{1/2}$

SOLUTION: **a.** Since

$$p_n = \frac{20p_{n-1} + 21/p_{n-1}^2}{21}, \quad \text{we have} \quad g(x) = \frac{20x + 21/x^2}{21} = \frac{20}{21}x + \frac{1}{x^2},$$

and $g'(x) = \dfrac{20}{21} - \dfrac{2}{x^3}$. Thus, $g'\left(21^{1/3}\right) = \dfrac{20}{21} - \dfrac{2}{21} = \dfrac{6}{7} \approx 0.857$.

b. Since

$$p_n = p_{n-1} - \frac{p_{n-1}^3 - 21}{3p_{n-1}^2}, \quad \text{we have} \quad g(x) = x - \frac{x^3 - 21}{3x^2} = x - \frac{1}{3}x + \frac{7}{x^2} = \frac{2}{3}x + \frac{7}{x^2}$$

and $g'(x) = \dfrac{2}{3} - \dfrac{7}{x^3}$. Thus, $g'\left(21^{1/3}\right) = \dfrac{2}{3} - \dfrac{1}{3} = \dfrac{1}{3} = 0.33\overline{3}$.

c. Since

$$p_n = p_{n-1} - \frac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21},$$

we have

$$g(x) = x - \frac{x^4 - 21x}{x^2 - 21} = \frac{x^3 - 21x - x^4 + 21x}{x^2 - 21} = \frac{x^3 - x^4}{x^2 - 21}$$

and

$$\begin{aligned} g'(x) =& \frac{\left(x^2-21\right)\left(3x^2-4x^3\right) - \left(x^3-x^4\right)2x}{\left(x^2-21\right)^2} = \frac{3x^4 - 63x^2 - 4x^5 + 84x^3 - 2x^4 + 2x^5}{\left(x^2-21\right)^2} \\ =& \frac{-2x^5 + x^4 + 84x^3 - 63x^2}{\left(x^2-21\right)^2}. \end{aligned}$$

Thus $g'\left(21^{1/3}\right) \approx 5.706 > 1$.

d. Since

$$p_n = \left(\frac{21}{p_{n-1}}\right)^{1/2}, \quad \text{we have} \quad g(x) = \left(\frac{21}{x}\right)^{1/2} = \frac{\sqrt{21}}{x^{1/2}},$$

and $g'(x) = \dfrac{-\sqrt{21}}{2x^{3/2}}$. Thus, $g'\left(21^{1/3}\right) = -\dfrac{1}{2}$.

The order of convergence would likely be (b), (d), (a). Choice (c) will not likely converge.

9. Use a fixed-point iteration method to determine an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}.

SOLUTION: As always with fixed-point iteration, the trick is to choose the fixed-point problem that will produce rapid convergence.

Recalling the solution to Exercise 12 in Section 2.1, we need to convert the root-finding problem $f(x) = x^2 - 3$ into a fixed-point problem. One successful solution is to write

$$0 = x^2 - 3 \quad \text{as} \quad x = \frac{3}{x},$$

then add x to both sides of the latter equation and divide by 2. This gives $g(x) = 0.5\left(x + \frac{3}{x}\right)$, and for $p_0 = 1.0$, we have $\sqrt{3} \approx p_4 = 1.73205$.

12. **c.** Determine a fixed-point function g and an appropriate interval that produces an approximation to a positive solution of $3x^2 - e^x = 0$ that is accurate to within 10^{-5}.

SOLUTION: There are numerous possibilities:

For $g(x) = \sqrt{\frac{1}{3}e^x}$ on $[0, 1]$ with $p_0 = 1$, we have $p_{12} = 0.910015$.

For $g(x) = \ln 3x^2$ on $[3, 4]$ with $p_0 = 4$, we have $p_{16} = 3.733090$.

14. Use a fixed-point iteration method to determine a solution accurate to within 10^{-4} for $x = \tan x$, for x in $[4, 5]$.

SOLUTION: Using $g(x) = \tan x$ and $p_0 = 4$ gives $p_1 = g(p_0) \approx 1.158$, which is not in the interval $[4, 5]$. So we need a different fixed-point function. If we note that $x = \tan x$ implies that

$$\frac{1}{x} = \frac{1}{\tan x} \quad \text{and define} \quad g(x) = x + \frac{1}{\tan x} - \frac{1}{x}$$

we obtain, again with $p_0 = 4$:

$$p_1 \approx 4.61369, \quad p_2 = 4.49596, \quad p_3 = 4.49341 \quad \text{and} \quad p_4 = 4.49341.$$

Because p_3 and p_4 agree to five decimal places it is reasonable to assume that these values are sufficiently accurate.

18. **a.** Show that Theorem 2.3 is true if $|g'(x)| \le k$ is replaced by the statement "$g'(x) \le k < 1$, for all $x \in [a, b]$".

b. Show that Theorem 2.4 may not hold when $|g'(x)| \le k$ is replaced by the statement "$g'(x) \le k < 1$, for all $x \in [a, b]$".

SOLUTION: **a.** The proof of existence is unchanged. For uniqueness, suppose p and q are fixed points in $[a, b]$ with $p \neq q$. By the Mean Value Theorem, a number ξ in (a, b) exists with

$$p - q = g(p) - g(q) = g'(\xi)(p - q) \leq k(p - q) < p - q,$$

giving the same contradiction as in Theorem 2.3.

b. For Theorem 2.4, consider $g(x) = 1 - x^2$ on $[0, 1]$. The function g has the unique fixed point $p = \frac{1}{2}\left(-1 + \sqrt{5}\right)$. With $p_0 = 0.7$, the sequence eventually alternates between numbers close to 0 and to 1, so there is no convergence.

19. **a.** Use Theorem 2.4 to show that the sequence

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}$$

converges for any $x_0 > 0$.

b. Show that if $0 < x_0 < \sqrt{2}$, then $x_1 > \sqrt{2}$.

c. Show that the sequence in (a) converges for every $x_0 > 0$.

SOLUTION: **a.** First let $g(x) = x/2 + 1/x$. For $x \neq 0$, we have $g'(x) = 1/2 - 1/x^2$. If $x > \sqrt{2}$, then $1/x^2 < 1/2$, so $g'(x) > 0$. Also, $g\left(\sqrt{2}\right) = \sqrt{2}$.

Suppose, as is the assumption given in part (a), that $x_0 > \sqrt{2}$. Then

$$x_1 - \sqrt{2} = g(x_0) - g\left(\sqrt{2}\right) = g'(\xi)\left(x_0 - \sqrt{2}\right),$$

where $\sqrt{2} < \xi < x_0$. Thus, $x_1 - \sqrt{2} > 0$ and $x_1 > \sqrt{2}$. Further,

$$x_1 = \frac{x_0}{2} + \frac{1}{x_0} < \frac{x_0}{2} + \frac{1}{\sqrt{2}} = \frac{x_0 + \sqrt{2}}{2},$$

and $\sqrt{2} < x_1 < x_0$. By an inductive argument, we have

$$\sqrt{2} < x_{m+1} < x_m < \ldots < x_0.$$

Thus, $\{x_m\}$ is a decreasing sequence that has a lower bound and must therefore converge. Suppose $p = \lim_{m\to\infty} x_m$. Then

$$p = \lim_{m\to\infty}\left(\frac{x_{m-1}}{2} + \frac{1}{x_{m-1}}\right) = \frac{p}{2} + \frac{1}{p}.$$

Thus

$$p = \frac{p}{2} + \frac{1}{p}, \quad \text{which implies that} \quad p^2 = 2,$$

so $p = \pm\sqrt{2}$. Since $x_m > \sqrt{2}$ for all m, $\lim_{m\to\infty} x_m = \sqrt{2}$.

b. Consider the situation when $0 < x_0 < \sqrt{2}$, which is the situation in part (b). Then we have

$$0 < \left(x_0 - \sqrt{2}\right)^2 = x_0^2 - 2x_0\sqrt{2} + 2,$$

so

$$2x_0\sqrt{2} < x_0^2 + 2 \quad \text{and} \quad \sqrt{2} < \frac{x_0}{2} + \frac{1}{x_0} = x_1.$$

c. To complete the problem, we consider the three possibilities for $x_0 > 0$.
Case 1: $x_0 > \sqrt{2}$, which by part (a) implies that $\lim_{m\to\infty} x_m = \sqrt{2}$.
Case 2: $x_0 = \sqrt{2}$, which implies that $x_m = \sqrt{2}$ for all m and that $\lim_{m\to\infty} x_m = \sqrt{2}$.
Case 3: $0 < x_0 < \sqrt{2}$, which implies that $\sqrt{2} < x_1$ by part (b). Thus

$$0 < x_0 < \sqrt{2} < x_{m+1} < x_m < \ldots < x_1 \quad \text{and} \quad \lim_{m\to\infty} x_m = \sqrt{2}.$$

In any situation, the sequence converges to $\sqrt{2}$, and rapidly, as we will discover in the Section 2.3.

24. Suppose that the function g has a fixed-point at p, that $g \in C[a, b]$, and that g' exists in (a, b). Show that if $|g'(p)| > 1$, then the fixed-point sequence will fail to converge for any initial choice of p_0, except if $p_n = p$ for some value of n.

SOLUTION: Since g' is continuous at p and $|g'(p)| > 1$, by letting $\epsilon = |g'(p)| - 1$ there exists a number $\delta > 0$ such that

$$|g'(x) - g'(p)| < \varepsilon = |g'(p)| - 1,$$

whenever $0 < |x - p| < \delta$. Since

$$|g'(x) - g'(p)| \geq |g'(p)| - |g'(x)|,$$

for any x satisfying $0 < |x - p| < \delta$, we have

$$|g'(x)| \geq |g'(p)| - |g'(x) - g'(p)| > |g'(p)| - (|g'(p)| - 1) = 1.$$

If p_0 is chosen so that $0 < |p - p_0| < \delta$, we have by the Mean Value Theorem that

$$|p_1 - p| = |g(p_0) - g(p)| = |g'(\xi)||p_0 - p|,$$

for some ξ between p_0 and p. Thus, $0 < |p - \xi| < \delta$ and

$$|p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|.$$

This means that when an approximation gets close to p, but is not equal to p, the succeeding terms of the sequence move away from p. So the sequence cannot converge to p.

Exercise Set 2.3, page 75

1. Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.

SOLUTION: Let $f(x) = x^2 - 6$. Then $f'(x) = 2x$, and Newton's method becomes

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} = p_{n-1} - \frac{p_{n-1}^2 - 6}{2p_{n-1}}.$$

With $p_0 = 1$, we have

$$p_1 = p_0 - \frac{p_0^2 - 6}{2p_0} = 1 - \frac{1 - 6}{2} = 1 + 2.5 = 3.5$$

and

$$p_2 = p_1 - \frac{p_1^2 - 6}{2p_1} = 3.5 - \frac{3.5^2 - 6}{2(3.5)} = 2.60714.$$

3. Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, find p_3 for **(a)** the Secant method and **(b)** the method of False Position.

c. Which method gives better results?

SOLUTION: The formula for both the Secant method and the method of False Position is

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$

a. The Secant method:

With $p_0 = 3$ and $p_1 = 2$, we have $f(p_0) = 9 - 6 = 3$ and $f(p_1) = 4 - 6 = -2$. The Secant method gives

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 2 - \frac{(-2)(2-3)}{-2-3} = 2 - \frac{2}{-5} = 2.4$$

and $f(p_2) = 2.4^2 - 6 = -0.24$. Then we have

$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)} = 2.4 - \frac{(-0.24)(2.4-2)}{(-0.24 - (-2)} = 2.4 - \frac{-0.096}{1.76} = 2.45454.$$

b. The method of False Position:

With $p_0 = 3$ and $p_1 = 2$, we have $f(p_0) = 3$ and $f(p_1) = -2$. As in the Secant method (part (a)), $p_2 = 2.4$ and $f(p_2) = -0.24$. Since $f(p_1) < 0$ and $f(p_2) < 0$, the method of False Position requires a reassignment of p_1. Then p_1 is changed to p_0 so that $p_1 = 3$, with $f(p_1) = 3$, and $p_2 = 2.4$, with $f(p_2) = -0.24$. We calculate p_3 by

$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)} = 2.4 - \frac{(-0.24)(2.4-3)}{-0.24-3} = 2.4 - \frac{0.144}{-3.24} = 2.44444.$$

c. Since $\sqrt{6} \approx 2.44949$, the accuracy of the approximations is the same. Continuing to more approximations would show that the Secant method is better.

5. **c.** Apply Newton's method to find a solution to $x - \cos x = 0$ in the interval $[0, \pi/2]$ that is accurate to within 10^{-4}.

SOLUTION: With $f(x) = x - \cos x$, we have $f'(x) = 1 + \sin x$, and the sequence generated by Newton's method is

$$p_n = p_{n-1} - \frac{p_{n-1} - \cos p_{n-1}}{1 + \sin p_{n-1}}.$$

For $p_0 = 0$, we have $p_1 = 1$, $p_2 = 0.75036$, $p_3 = 0.73911$, and $p_4 = 0.73909$.

7. **c.** Apply the Secant method to find a solution to $x - \cos x = 0$ in the interval $[0, \pi/2]$ that is accurate to within 10^{-4}.

SOLUTION: The Secant method approximations are generated by the sequence

$$p_n = p_{n-1} - \frac{(p_{n-1} - \cos p_{n-1})(p_{n-1} - p_{n-2})}{(p_{n-1} - \cos p_{n-1}) - (p_{n-2} - \cos p_{n-2})}.$$

Using the endpoints of the intervals as p_0 and p_1, we have the entries in the following table.

n	p_n
0	0
1	1.5707963
2	0.6110155
3	0.7232695
4	0.7395671
5	0.7390834
6	0.7390851

9. c. Apply the method of False Position to find a solution to $x - \cos x = 0$ in the interval $[0, \pi/2]$ that is accurate to within 10^{-4}.

SOLUTION: The method of False Position approximations are generated using this same formula as in Exercise 7, but incorporates the additional bracketing test. Using the endpoints of the intervals as p_0 and p_1, we have the entries in the following table.

n	p_n
0	0
1	1.5707963
2	0.6110155
3	0.7232695
4	0.7372659
5	0.7388778
6	0.7390615
7	0.7390825

13. Apply Newton's method to find a solution, accurate to within 10^{-4}, to the value of x that produces the closest point on the graph of $y = x^2$ to the point $(1, 0)$.

SOLUTION: The distance between an arbitrary point (x, x^2) on the graph of $y = x^2$ and the point $(1, 0)$ is

$$d(x) = \sqrt{(x-1)^2 + (x^2 - 0)^2} = \sqrt{x^4 + x^2 - 2x + 1}.$$

Because a derivative is needed to find the critical points of d, it is easier to work with the square of this function,

$$f(x) = [d(x)]^2 = x^4 + x^2 - 2x + 1,$$

whose minimum will occur at the same value of x as the minimum of $d(x)$. To minimize $f(x)$ we need x so that $0 = f'(x) = 4x^3 + 2x - 2$.

Applying Newton's method to find the root of this equation with $p_0 = 1$ gives $p_5 = 0.589755$. The point on the graph of $y = x^2$ that is closest to $(1, 0)$ has the approximate coordinates $(0.589755, 0.347811)$.

16. Use Newton's method to solve for roots of

$$0 = \frac{1}{2} + \frac{1}{4}x^2 - x\sin x - \frac{1}{2}\cos 2x.$$

SOLUTION: Newton's method with $p_0 = \frac{\pi}{2}$ gives $p_{15} = 1.895488$ and with $p_0 = 5\pi$ gives $p_{19} = 1.895489$. With $p_0 = 10\pi$, the sequence does not converge in 200 iterations.

The results do not indicate the fast convergence usually associated with Newton's method because the function and its derivative have the same roots. As we approach a root, we are dividing by numbers with small magnitude, which increases the round-off error.

19. Explain why the iteration equation for the Secant method should not be used in the algebraically equivalent form

$$p_n = \frac{f(p_{n-1})p_{n-2} - f(p_{n-2})p_{n-1}}{f(p_{n-1}) - f(p_{n-2})}.$$

SOLUTION: This formula incorporates the subtraction of nearly equal numbers in both the numerator and denominator when p_{n-1} and p_{n-2} are nearly equal. The form given in the Secant Algorithm subtracts a correction from a result that should dominate the calculations. This is always the preferred approach.

22. Use Maple to determine how many iterations of Newton's method with $p_0 = \pi/4$ are needed to find a root of $f(x) = \cos x - x$ to within 10^{-100}.

SOLUTION: We first define $f(x)$ and $f'(x)$ with

$f := x- > cos(x) - x$

$$f := x \to \cos(x) - x$$

and

$fp := x- > (D)(f)(x)$

$$fp := x \to -\sin(x) - 1$$

We wish to use 100-digit rounding arithmetic so we set

$Digits := 100; p0 := Pi/4$

$$Digits := 100$$

$$p0 := \frac{1}{4}\pi$$

for n from 1 *to* 7 *do*

$p1 :=$ *evalf*$(p0 - f(p0)/fp(p0))$

$err := abs(p1 - p0)$

$p0 := p1$

od

This gives

$$p_7 = .7390851332151606416553120876738734040134117589007574649656806357732846548835475945993761069317665319,$$

which is accurate to 10^{-100}.

23. The function defined by $f(x) = \ln\left(x^2+1\right) - e^{0.4x}\cos \pi x$ has an infinite number of zeros.

a. Approximate the only negative zero to within 10^{-6}.

b. Approximate the four smallest positive zeros to within 10^{-6}.

c. Find an initial approximation for the nth smallest positive zero.

d. Approximate the 25th smallest positive zero to within 10^{-6}.

SOLUTION: The key to this problem is recognizing the behavior of $e^{0.4x}$. When x is negative, this term goes to zero, so $f(x)$ is dominated by $\ln\left(x^2+1\right)$. However, when x is positive, $e^{0.4x}$ dominates the calculations, and $f(x)$ will be zero approximately when this term makes no contribution; that is, when $\cos \pi x = 0$. This occurs when $x = n/2$ for a positive integer n. Using this information to determine initial approximations produces the following results:

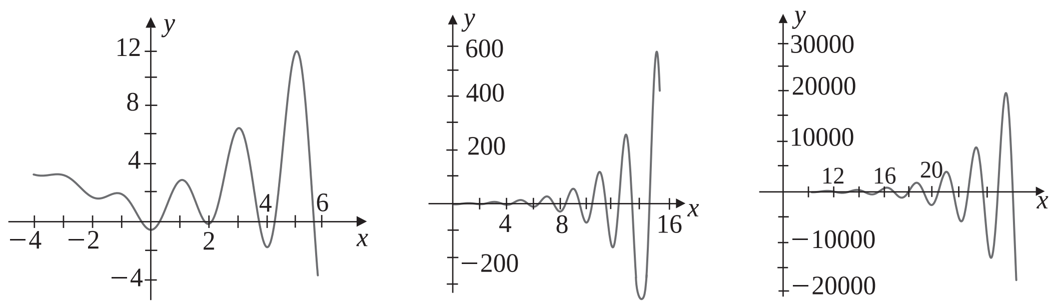
a. We can use $p_0 = -0.5$ to find the sufficiently accurate $p_3 = -0.4341431$.

b. We can use: $p_0 = 0.5$ to give $p_3 = 0.4506567$; $p_0 = 1.5$ to give $p_3 = 1.7447381$; $p_0 = 2.5$ to give $p_5 = 2.2383198$; and $p_0 = 3.5$ to give $p_4 = 3.7090412$.

c. In general, a reasonable initial approximation for the nth positive root is $n - 0.5$.

d. Let $p_0 = 24.5$. A sufficiently accurate approximation to the 25th smallest positive zero is $p_2 = 24.4998870$.

Graphs for various parts of the region are shown below.



26. Determine the minimal annual interest rate i at which an amount $P = \$1500$ per month can be invested to accumulate an amount $A = \$750,000$ at the end of 20 years based on the annuity due equation

$$A = \frac{P}{i}\left[(1+i)^n - 1\right].$$

SOLUTION: This is simply a root-finding problem where the function is given by

$$f(i) = A - \frac{P}{i}\left[(1+i)^n - 1\right] = 750000 - \frac{1500}{(i/12)}\left[(1+i/12)^{(12)(20)} - 1\right].$$

Notice that n and i have been adjusted because the payments are made monthly rather than yearly. The approximate solution to this equation can be found by any method in this section. Newton's method is a bit cumbersome for this problem, since the derivative of f is complicated. The Secant method would be a likely choice. The minimal annual interest is approximately 6.67%.

28. A drug administered to a patient produces a concentration in the blood stream given by $c(t) = Ate^{-t/3}$ mg/mL, t hours after A units have been administered. The maximum safe concentration is 1 mg/mL.

a. What amount should be injected to reach this safe level, and when does this occur?

b. When should an additional amount be administered, if it is administered when the level drops to 0.25 mg/mL?

c. Assuming 75% of the original amount is administered in the second injection, when should a third injection be given?

SOLUTION: **a.** The maximum concentration occurs when

$$0 = c'(t) = A\left(1 - \frac{t}{3}\right)e^{-t/3}.$$

This happens when $t = 3$ hours, and since the concentration at this time will be $c(3) = 3Ae^{-1}$, we need to administer $A = \frac{1}{3}e$ units.

b. We need to determine t so that

$$0.25 = c(t) = \left(\frac{1}{3}e\right)te^{-t/3}.$$

This occurs when t is 11 hours and 5 minutes; that is, when $t = 11.08\overline{3}$ hours.

c. We need to find t so that

$$0.25 = c(t) = \left(\frac{1}{3}e\right)te^{-t/3} + 0.75\left(\frac{1}{3}e\right)(t - 11.08\overline{3})e^{-(t-11.08\overline{3})/3}.$$

This occurs after 21 hours and 14 minutes.

29. Let $f(x) = 3^{3x+1} - 7 \cdot 5^{2x}$.

a. Use the Maple commands *solve* and *fsolve* to try to find all roots of f.

b. Plot $f(x)$ to find initial approximations to roots of f.

c. Use Newton's method to find the zeros of f to within 10^{-16}.

d. Find the exact solutions of $f(x) = 0$ algebraically.

SOLUTION: **a.** First define the function by

$f := x- > 3^{3x+1} - 7 \cdot 5^{2x}$

$$f := x \to 3^{(3x+1)} - 7\,5^{2x}$$

solve$(f(x) = 0, x)$

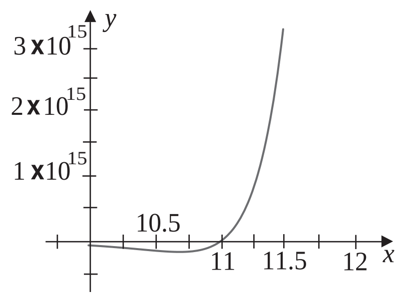
$$-\frac{\ln(3/7)}{\ln(27/25)}$$

fsolve$(f(x) = 0, x)$

$$\text{fsolve}(3^{(3x+1)} - 7\,5^{(2x)} = 0, x)$$

The procedure *solve* gives the exact solution, and *fsolve* fails because the negative x-axis is an asymptote for the graph of $f(x)$.

b. Using the Maple command *plot*$(\{f(x)\}, x = 9.5..11.5)$ produces the following graph.



c. Define $f'(x)$ using
$fp := x- > (D)(f)(x)$

$$fp := x \to 3\,3^{(3x+1)}\ln(3) - 14\,5^{(2x)}\ln(5)$$

Digits $:= 18; p0 := 11$

$$\text{Digits} := 18$$
$$p0 := 11$$

for i from 1 *to* 5 *do*
$p1 :=$ *evalf*$(p0 - f(p0)/fp(p0))$
$err := abs(p1 - p0)$
$p0 := p1$
od

The results are given in the following table.

i	p_i	$\lvert p_i - p_{i-1}\rvert$
1	11.0097380401552503	0.0097380401552503
2	11.0094389359662827	0.0002991041889676
3	11.0094386442684488	$0.2916978339\ 10^{-6}$
4	11.0094386442681716	$0.2772\ 10^{-12}$
5	11.0094386442681716	0

d. We have $3^{3x+1} = 7 \cdot 5^{2x}$. Taking the natural logarithm of both sides gives

$$(3x+1)\ln 3 = \ln 7 + 2x\ln 5.$$

Thus

$$3x\ln 3 - 2x\ln 5 = \ln 7 - \ln 3, \quad x(3\ln 3 - 2\ln 5) = \ln\frac{7}{3},$$

and

$$x = \frac{\ln 7/3}{\ln 27 - \ln 25} = \frac{\ln 7/3}{\ln 27/25} = -\frac{\ln 3/7}{\ln 27/25}.$$

This agrees with part (a).

Exercise Set 2.4, page 85

1. a. Use Newton's method to find a solution accurate to within 10^{-5} for $x^2 - 2xe^{-x} + e^{-2x} = 0$, where $0 \le x \le 1$.

SOLUTION: Since

$$f(x) = x^2 - 2xe^{-x} + e^{-2x} \quad \text{and} \quad f'(x) = 2x - 2e^{-x} + 2xe^{-x} - 2e^{-2x},$$

the iteration formula is

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} = p_{n-1} - \frac{p_{n-1}^2 - 2p_{n-1}e^{-p_{n-1}} + e^{-2p_{n-1}}}{2p_{n-1} - 2e^{-p_{n-1}} + 2p_{n-1}e^{-p_{n-1}} - 2e^{-2p_{n-1}}}.$$

With $p_0 = 0.5$, we have

$$p_1 = 0.5 - (0.01134878)/(-0.3422895) = 0.5331555.$$

Continuing in this manner, $p_{13} = 0.567135$ is accurate to within 10^{-5}.

3. a. Repeat Exercise 1(a) using the modified Newton-Raphson method described in Eq. (2.13). Is there an improvement in speed or accuracy over Exercise 1?

SOLUTION: Since

$$\begin{aligned} f(x) =& x^2 - 2xe^{-x} + e^{-2x}, \\ f'(x) =& 2x - 2e^{-x} + 2xe^{-x} - 2e^{-2x}, \end{aligned}$$

and

$$f''(x) = 2 + 4e^{-x} - 2xe^{-x} + 4e^{-2x},$$

the iteration formula is

$$p_n = p_{n-1} - \frac{f(p_{n-1})f'(p_{n-1})}{[f'(p_{n-1})]^2 - f(p_{n-1})f''(p_{n-1})}.$$

With $p_0 = 0.5$, we have $f(p_0) = 0.011348781$, $f'(p_0) = -0.342289542$, $f''(p_0) = 5.291109744$ and

$$p_1 = 0.5 - \frac{(0.01134878)(-0.342289542)}{(-0.342289542)^2 - (0.011348781)(5.291109744)} = 0.5680137.$$

Continuing in this manner, $p_3 = 0.567143$ is accurate to within 10^{-5}, which is considerably better than in Exercise 1.

6. a. Show that the sequence $p_n = 1/n$ converges linearly to $p = 0$, and determine the number of terms required to have $|p_n - p| < 5 \times 10^{-2}$.

SOLUTION: First note that $\lim_{n\to\infty} \frac{1}{n} = 0$. Since

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$

the convergence is linear. To have $|p_n - p| < 5 \times 10^{-2}$, we need $1/n < 0.05$, which implies that $n > 20$.

8. Show that:

a. The sequence $p_n = 10^{-2^n}$ converges quadratically to zero;

b. The sequence $p_n = 10^{-n^k}$ does not converge to zero quadratically, regardless of the size of $k > 1$.

SOLUTION:

a. Since

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{\left(10^{-2^n}\right)^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2\cdot 2^n}} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1,$$

the sequence is quadratically convergent.

b. For any $k > 1$,

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{\left(10^{-n^k}\right)^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}} = \lim_{n\to\infty} 10^{2n^k - (n+1)^k}$$

diverges. So the sequence $p_n = 10^{-n^k}$ does not converge quadratically for any $k > 1$.

10. Show that the fixed-point method

$$g(x) = x - \frac{mf(x)}{f'(x)}$$

has $g'(p) = 0$, if p is a zero of f of multiplicity m.

SOLUTION: If f has a zero of multiplicity m at p, then a function q exists with

$$f(x) = (x-p)^m q(x), \quad \text{where} \quad \lim_{x\to p} q(x) \neq 0.$$

Since

$$f'(x) = m(x-p)^{m-1}q(x) + (x-p)^m q'(x),$$

we have

$$g(x) = x - \frac{mf(x)}{f'(x)} = x - \frac{m(x-p)^m q(x)}{m(x-p)^{m-1}q(x) + (x-p)^m q'(x)},$$

which reduces to

$$g(x) = x - \frac{m(x-p)q(x)}{mq(x) + (x-p)q'(x)}.$$

Differentiating this expression and evaluating at $x = p$ gives

$$g'(p) = 1 - \frac{mq(p)[mq(p)]}{[mq(p)]^2} = 0.$$

If f''' is continuous, Theorem 2.9 implies that this sequence produces quadratic convergence once we are close enough to the solution p.

12. Suppose that f has m continuous derivatives. Show that f has a zero of multiplicity m at p if and only if

$$0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p), \quad \text{but} \quad f^{(m)}(p) \neq 0.$$

SOLUTION: If f has a zero of multiplicity m at p, then f can be written as

$$f(x) = (x-p)^m q(x), \quad \text{for } x \neq p, \text{ where } \quad \lim_{x \to p} q(x) \neq 0.$$

Thus

$$f'(x) = m(x-p)^{m-1}q(x) + (x-p)^m q'(x)$$

and $f'(p) = 0$. Also

$$f''(x) = m(m-1)(x-p)^{m-2}q(x) + 2m(x-p)^{m-1}q'(x) + (x-p)^m q''(x)$$

and $f''(p) = 0$.

In general, for $k \leq m$,

$$\begin{aligned} f^{(k)}(x) &= \sum_{j=0}^{k} \binom{k}{j} \frac{d^j (x-p)^m}{dx^j} q^{(k-j)}(x) \\ &= \sum_{j=0}^{k} \binom{k}{j} m(m-1)\cdots(m-j+1)(x-p)^{m-j} q^{(k-j)}(x). \end{aligned}$$

Thus, for $0 \leq k \leq m-1$, we have $f^{(k)}(p) = 0$, but

$$f^{(m)}(p) = m! \lim_{x \to p} q(x) \neq 0.$$

Conversely, suppose that $f(p) = f'(p) = \ldots = f^{(m-1)}(p) = 0$ and $f^{(m)}(p) \neq 0$. Consider the $(m-1)$th Taylor polynomial of f expanded about p :

$$\begin{aligned} f(x) =& f(p) + f'(p)(x-p) + \ldots + \frac{f^{(m-1)}(p)(x-p)^{m-1}}{(m-1)!} + \frac{f^{(m)}(\xi(x))(x-p)^m}{m!} \\ =& (x-p)^m \frac{f^{(m)}(\xi(x))}{m!}, \end{aligned}$$

where $\xi(x)$ is between x and p. Since $f^{(m)}$ is continuous, let

$$q(x) = \frac{f^{(m)}(\xi(x))}{m!}.$$

Then $f(x) = (x-p)^m q(x)$ and

$$\lim_{x \to p} q(x) = \frac{f^{(m)}(p)}{m!} \neq 0.$$

So p is a zero of multiplicity m.

14. Show that the Secant method converges of order α, where $\alpha = \left(1+\sqrt{5}\right)/2$, the golden ratio.

SOLUTION: Let $e_n = p_n - p$. If

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda > 0,$$

then for sufficiently large values of n, $|e_{n+1}| \approx \lambda |e_n|^\alpha$. Thus

$$|e_n| \approx \lambda |e_{n-1}|^\alpha \quad \text{and} \quad |e_{n-1}| \approx \lambda^{-1/\alpha} |e_n|^{1/\alpha}.$$

The hypothesis that for some constant C and sufficiently large n we have $|p_{n+1}-p| \approx C|p_n - p|\ |p_{n-1}-p|$, gives

$$\lambda|e_n|^\alpha \approx C|e_n|\lambda^{-1/\alpha}|e_n|^{1/\alpha}, \quad \text{so} \quad |e_n|^\alpha \approx C\lambda^{-1/\alpha-1}|e_n|^{1+1/\alpha}.$$

Since the powers of $|e_n|$ must agree,

$$\alpha = 1 + 1/\alpha \quad \text{and} \quad \alpha = \frac{1+\sqrt{5}}{2}.$$

This number, the *Golden Ratio*, appears in numerous situations in mathematics and in art.

Exercise Set 2.5, page 90

2. Apply Newton's method to approximate a root of

$$f(x) = e^{6x} + 3(\ln 2)^2 e^{2x} - \ln 8e^{4x} - (\ln 2)^3 = 0.$$

Generate terms until $|p_{n+1} - p_n| < 0.0002$, and construct the Aitken's Δ^2 sequence $\{\hat{p}_n\}$.

SOLUTION: Applying Newton's method with $p_0 = 0$ requires finding $p_{16} = -0.182888$. For the Aitken's Δ^2 sequence, we have sufficient accuracy with $\hat{p}_6 = -0.183387$. Newton's method fails to converge quadratically because there is a multiple root.

3. Let $g(x) = \cos(x-1)$ and $p_0^{(0)} = 2$. Use Steffensen's method to find $p_0^{(1)}$.

SOLUTION: With $g(x) = \cos(x-1)$ and $p_0^{(0)} = 2$, we have

$$p_1^{(0)} = g\left(p_0^{(0)}\right) = \cos(2-1) = \cos 1 = 0.5403023$$

and

$$p_2^{(0)} = g\left(p_1^{(0)}\right) = \cos(0.5403023 - 1) = 0.8961867.$$

Thus

$$\begin{aligned} p_0^{(1)} &= p_0^{(0)} - \frac{\left(p_1^{(0)} - p_0^{(0)}\right)^2}{p_2^{(0)} - 2p_1^{(0)} - 2p_1^{(0)} + p_0^{(0)}} \\ &= 2 - \frac{(0.5403023-2)^2}{0.8961867 - 2(0.5403023) + 2} = 2 - 1.173573 = 0.826427. \end{aligned}$$

5. Steffensen's method is applied to a function $g(x)$ using $p_0^{(0)} = 1$ and $p_2^{(0)} = 3$ to obtain $p_0^{(1)} = 0.75$. What could $p_1^{(0)}$ be?

SOLUTION: Steffensen's method uses the formula

$$p_1^{(0)} = p_0^{(0)} - \frac{\left(p_1^{(0)} - p_0^{(0)}\right)^2}{p_2^{(0)} - 2p_1^{(0)} + p_0^{(0)}}.$$

Substituting for $p_0^{(0)}$, $p_2^{(0)}$, and $p_0^{(1)}$ gives

$$0.75 = 1 - \frac{\left(p_1^{(0)} - 1\right)^2}{3 - 2p_1^{(0)} + 1}, \quad \text{that is,} \quad 0.25 = \frac{\left(p_1^{(0)} - 1\right)^2}{4 - 2p_1^{(0)}}.$$

Thus

$$1 - \frac{1}{2}p_1^{(0)} = \left(p_1^{(0)}\right)^2 - 2p_1^{(0)} + 1, \quad \text{so} \quad 0 = \left(p_1^{(0)}\right)^2 - 1.5p_1^{(0)},$$

and $p_1^{(0)} = 1.5$ or $p_1^{(0)} = 0$.

11. b. Use Steffensen's method to approximate the solution to within 10^{-5} of $x = 0.5(\sin x + \cos x)$, where $g(x) = 0.5(\sin x + \cos x)$.

SOLUTION: With $g(x) = 0.5(\sin x + \cos x)$, we have

$$\begin{aligned}
p_0^{(0)} &= 0,\ p_1^{(0)} = g(0) = 0.5, \\
p_2^{(0)} &= g(0.5) = 0.5(\sin 0.5 + \cos 0.5) = 0.678504051, \\
p_0^{(1)} &= p_0^{(0)} - \frac{\left(p_1^{(0)} - p_0^{(0)}\right)^2}{p_2^{(0)} - 2p_1^{(0)} + p_0^{(0)}} = 0.777614774, \\
p_1^{(1)} &= g\left(p_0^{(1)}\right) = 0.707085363, \\
p_2^{(1)} &= g\left(p_1^{(1)}\right) = 0.704939584, \\
p_0^{(2)} &= p_0^{(1)} - \frac{\left(p_1^{(1)} - p_0^{(1)}\right)^2}{p_2^{(1)} - 2p_1^{(1)} + p_0^{(1)}} = 0.704872252, \\
p_1^{(2)} &= g\left(p_0^{(2)}\right) = 0.704815431, \\
p_2^{(2)} &= g\left(p_1^{(2)}\right) = 0.704812197, \\
p_0^{(3)} &= p_0^{(2)} = \frac{\left(p_1^{(2)} - p_0^{(2)}\right)^2}{p_2^{(2)} - 2p_1^{(2)} + p_0^{(2)}} = 0.704812002, \\
p_1^{(3)} &= g\left(p_0^{(3)}\right) = 0.704812002,
\end{aligned}$$

and

$$p_2^{(3)} = g\left(p_1^{(3)}\right) = 0.704812197.$$

Since $p_2^{(3)}$, $p_1^{(3)}$, and $p_0^{(3)}$ all agree to within 10^{-5}, we accept $p_2^{(3)} = 0.704812197$ as an answer that is accurate to within 10^{-5}.

14. a. Show that a sequence $\{p_n\}$ that converges to p with order $\alpha > 1$ converges superlinearly to p.

b. Show that the sequence $p_n = \dfrac{1}{n^n}$ converges superlinearly to 0, but does not converge of order α for any $\alpha > 1$.

SOLUTION: Since $\{p_n\}$ converges to p with order $\alpha > 1$, a positive constant λ exists with

$$\lambda = \lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha}.$$

Hence

$$\lim_{n\to\infty}\left|\frac{p_{n+1}-p}{p_n-p}\right| = \lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^\alpha}\cdot|p_n-p|^{\alpha-1} = \lambda\cdot 0 = 0 \quad\text{and}\quad \lim_{n\to\infty}\frac{p_{n+1}-p}{p_n-p} = 0.$$

This implies that $\{p_n\}$ that converges superlinearly to p.

b. The sequence converges $p_n = \dfrac{1}{n^n}$ superlinearly to zero because

$$\begin{aligned}\lim_{n\to\infty}\frac{1/(n+1)^{(n+1)}}{1/n^n} &= \lim_{n\to\infty}\frac{n^n}{(n+1)^{(n+1)}}\\ &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\frac{1}{n+1}\\ &= \lim_{n\to\infty}\left(\frac{1}{(1+1/n)^n}\right)\frac{1}{n+1} = \frac{1}{e}\cdot 0 = 0.\end{aligned}$$

However, for $\alpha > 1$, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1/(n+1)^{(n+1)}}{(1/n^n)^\alpha} &= \lim_{n\to\infty}\frac{n^{\alpha n}}{(n+1)^{(n+1)}}\\ &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\frac{n^{(\alpha-1)n}}{n+1}\\ &= \lim_{n\to\infty}\left(\frac{1}{(1+1/n)^n}\right)\lim_{n\to\infty}\frac{n^{(\alpha-1)n}}{n+1} = \frac{1}{e}\cdot\infty = \infty.\end{aligned}$$

So the sequence does not converge of order α for any $\alpha > 1$.

17. Let $P_n(x)$ be the nth Taylor polynomial for $f(x) = e^x$ expanded about $x_0 = 0$.

a. For fixed x, show that $p_n = P_n(x)$ satisfies the hypotheses of Theorem 2.14.

b. Let $x = 1$, and use Aitken's Δ^2 method to generate the sequence $\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_8$.

c. Does Aitken's Δ^2 method accelerate the convergence in this situation?

SOLUTION: **a.** Since

$$p_n = P_n(x) = \sum_{k=0}^{n}\frac{1}{k!}x^k,$$

we have

$$p_n - p = P_n(x) - e^x = \frac{-e^\xi}{(n+1)!}x^{n+1},$$

where ξ is between 0 and x. Thus, $p_n - p \neq 0$, for all $n \geq 0$. Further,

$$\frac{p_{n+1}-p}{p_n-p} = \frac{\frac{-e^{\xi_1}}{(n+2)!}x^{n+2}}{\frac{-e^{\xi}}{(n+1)!}x^{n+1}} = \frac{e^{(\xi_1-\xi)}x}{n+2},$$

where ξ_1 is between 0 and 1. Thus

$$\lambda = \lim_{n\to\infty}\frac{e^{(\xi_1-\xi)}x}{n+2} = 0 < 1.$$

b. The sequence has the terms shown in the following tables.

n	0	1	2	3	4	5	6
p_n	1	2	2.5	$2.\overline{6}$	$2.708\overline{3}$	$2.71\overline{6}$	$2.7180\overline{5}$
$\hat{p}_n$	3	2.75	$2.7\overline{2}$	2.71875	$2.718\overline{3}$	2.7182870	2.7182823

n	7	8	9	10
p_n	2.7182539	2.7182787	2.7182815	2.7182818
$\hat{p}_n$	2.7182818	2.7182818		

c. Aitken's Δ^2 method gives quite an improvement for this problem. For example, $\hat{p}_6$ is accurate to within 5×10^{-7}. We need p_{10} to have this accuracy.

Exercise Set 2.6, page 100

2. b. Use Newton's method to approximate, to within 10^{-5}, the real zeros of

$$P(x) = x^4 - 2x^3 - 12x^2 + 16x - 40.$$

Then reduce the polynomial to lower degree, and determine any complex zeros.

SOLUTION: Applying Newton's method with $p_0 = 1$ gives the sufficiently accurate approximation $p_7 = -3.548233$. When $p_0 = 4$, we find another zero to be $p_5 = 4.381113$. If we divide $P(x)$ by

$$(x + 3.548233)(x - 4.381113) = x^2 - 0.832880x - 15.54521,$$

we find that

$$P(x) \approx \left(x^2 - 0.832880x - 15.54521\right)\left(x^2 - 1.16712x + 2.57315\right).$$

The complex roots of the quadratic on the right can be found by the quadratic formula and are approximately $0.58356 \pm 1.49419i$.

4. b. Use Müller's method to find the real and complex zeros of

$$P(x) = x^4 - 2x^3 - 12x^2 + 16x - 40.$$

SOLUTION: The following table lists the initial approximation and the roots. The first initial approximation was used because $f(0) = -40$, $f(1) = -37$, and $f(2) = -56$ implies that there is a minimum in $[0, 2]$. This is confirmed by the complex roots that are generated.

The second initial approximations are used to find the real root that is known to lie between 4 and 5, due to the fact that $f(4) = -40$ and $f(5) = 115$.

The third initial approximations are used to find the real root that is known to lie between -3 and -4, since $f(-3) = -61$ and $f(-4) = 88$.

p_0	p_1	p_2	Approximated Roots	Complex Conjugate Root
0	1	2	$p_7 = 0.583560 - 1.494188i$	$0.583560 + 1.494188i$
2	3	4	$p_6 = 4.381113$	
-2	-3	-4	$p_5 = -3.548233$	

5. b. Find the zeros and critical points of

$$f(x) = x^4 - 2x^3 - 5x^2 + 12x - 5,$$

and use this information to sketch the graph of f.

SOLUTION: There are at most four real zeros of f and $f(0) < 0$, $f(1) > 0$, and $f(2) < 0$. This, together with the fact that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$, implies that these zeros lie in the intervals $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, and $(2, \infty)$. Applying Newton's method for various initial approximations in these intervals gives the approximate zeros: 0.5798, 1.521, 2.332, and -2.432. To find the critical points, we need the zeros of

$$f'(x) = 4x^3 - 6x^2 - 10x + 12.$$

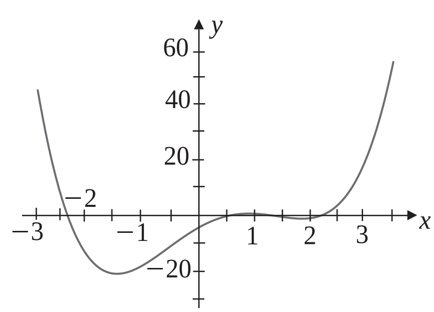
Since $x = 1$ is quite easily seen to be a zero of $f'(x)$, the cubic equation can be reduced to a quadratic to find the other two zeros: 2 and -1.5.

Since the quadratic formula applied to

$$0 = f''(x) = 12x^2 - 12x - 10$$

gives $x = 0.5 \pm \left(\sqrt{39}/6\right)$, we also have the points of inflection.

A sketch of the graph of f is given below.



9. Find a solution, accurate to within 10^{-4}, to the problem

$$600x^4 - 550x^3 + 200x^2 - 20x - 1 = 0, \quad \text{for } 0.1 \le x \le 1$$

by using the various methods in this chapter.

SOLUTION:

a. Bisection method: For $p_0 = 0.1$ and $p_1 = 1$, we have $p_{14} = 0.23233$.

b. Newton's method: For $p_0 = 0.55$, we have $p_6 = 0.23235$.

c. Secant method: For $p_0 = 0.1$ and $p_1 = 1$, we have $p_8 = 0.23235$.

d. Method of False Position: For $p_0 = 0.1$ and $p_1 = 1$, we have $p_{88} = 0.23025$.

e. Müller's method: For $p_0 = 0$, $p_1 = 0.25$, and $p_2 = 1$, we have $p_6 = 0.23235$.

Notice that the method of False Position for this problem was considerably less effective than both the Secant method and the Bisection method.

11. A can in the shape of a right circular cylinder must have a volume of 1000 cm^3. To form seals, the top and bottom must have a radius 0.25 cm more than the radius and the material for the side must be 0.25 cm longer than the circumference of the can. Minimize the amount of material that is required.

SOLUTION: Since the volume is given by

$$V = 1000 = \pi r^2 h,$$

we have $h = 1000/\left(\pi r^2\right)$. The amount of material required for the top of the can is $\pi(r + 0.25)^2$, and a similar amount is needed for the bottom. To construct the side of the can, the material needed is $(2\pi r + 0.25)h$. The total amount of material $M(r)$ is given by

$$M(r) = 2\pi(r + 0.25)^2 + (2\pi r + 0.25)h = 2\pi(r + 0.25)^2 + 2000/r + 250/\pi r^2.$$

Thus

$$M'(r) = 4\pi(r + 0.25) - 2000/r^2 - 500/(\pi r^3).$$

Solving $M'(r) = 0$ for r gives $r \approx 5.363858$. Evaluating $M(r)$ at this value of r gives the minimal material needed to construct the can:

$$M(5.363858) \approx 573.649 \text{ cm}^2.$$

12. Leonardo of Pisa (Fibonacci) found the base 60 approximation

$$1 + 22\left(\frac{1}{60}\right) + 7\left(\frac{1}{60}\right)^2 + 42\left(\frac{1}{60}\right)^3 + 33\left(\frac{1}{60}\right)^4 + 4\left(\frac{1}{60}\right)^5 + 40\left(\frac{1}{60}\right)^6$$

as a root of the equation

$$x^3 + 2x^2 + 10x = 20.$$

How accurate was his approximation?

SOLUTION: The decimal equivalent of Fibonacci's base 60 approximation is 1.3688081078532, and Newton's Method gives 1.36880810782137 with a tolerance of 10^{-16}. So Fibonacci's answer was correct to within 3.2×10^{-11}. This is the most accurate approximation to an irrational root of a cubic polynomial that is known to exist, at least in Europe, before the sixteenth century. Fibonacci probably learned the technique for approximating this root from the writings of the great Persian poet and mathematician Omar Khayyám.